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# Surface tension and SOS limit in the 2D Ising model

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**Abstract.** The Sherman theorem on closed paths for the two-dimensional Ising model on a square with + boundary conditions is generalised to arbitrary boundary conditions. The refinement allows a rigorous cluster expansion for boundary observables (in particular for the surface tension) in terms of open random trajectories on the lattice. As an application we discuss the grand canonical cluster expansion for the surface tension and prove its convergence to the canonical cluster expansion and to the SOS limit.

## 1. Introduction

The combinatorial approach to the solution of the two-dimensional Ising model in zero field is well known and has been considered in past years (Newell and Montroll 1953, Sherman 1960, Burgoyne 1963, Glasser 1970). The rigorous treatment of the combinatorics (for the case of open or periodic boundary conditions) for the free energy was given by Sherman (1963), who rigorously proved the Feynman–Kac conjecture on paths and graphs. The above theorem has been used more recently, in connection with the analysis of the absence of non-translationally invariant equilibrium states in the two-dimensional Ising model at low-temperature (Merlini 1978, 1981). A problem which is often discussed is the relation between the Ising model and the SOS (solid-on-solid) model in all dimensions. In particular one expects that in some two-dimensional models, the surface tension computed in the SOS limit coincides with the exact result for $T \leqslant T_c$ ($T_c$ being the critical temperature). Nevertheless, the fundamental reason for this is still unknown.

This problem was discussed recently and it was claimed that a direct analysis of the graphical structure associated with the system, in addition to an application of the law of large numbers, could be performed (Groeneweld 1982). On the other hand it may be that, even in two dimensions, the SOS limit for the surface tension yields the exact result for a restricted class of many-body interactions on the lattice and this problem is not related to the absence of non-translationally invariant equilibrium states in two dimensions.

Here we do not treat the general problem suggested by Groeneweld; we merely restrict ourselves to the 2D Ising model on a square lattice. For this model, we first reconsider the Sherman theorem on paths and graphs for an open square (established

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for + or equivalently for open boundary conditions by duality), and generalise it to arbitrary boundary conditions $b$ in § 2; the ratio between two partition functions with arbitrary conditions and the cluster expansion for observables at the boundary may then be given in terms of a well defined set of open and closed trajectories on the lattice. As an application we first analyse the cluster expansion for the grand canonical surface tension and show rigorously its convergence to that of the canonical one; moreover we check graphically its relation to the SOS limit in § 3 for $T \leqslant T_c$. In § 4 we give a proof that the grand canonical surface tension coincides with that of the SOS limit for $T \leqslant T_c$; the Onsager formula is also recovered within the combinatorial method we have considered. The coincidence of the SOS limit with the exact result is shown to be connected with the property of the eigenvalues of the matrix propagator for trajectories. The result is established for the two-dimensional Ising model; it is nevertheless expected that it may be extended by other methods to a more general class of two-dimensional models. The combinatorial method may further be useful in the analysis of some interesting calculations for the Ising model.

## 2. A refinement of the Sherman theorem on paths

Let $\Lambda \subset \mathbb{Z}^2$ be a finite rectangular box of $|\Lambda|$ sites. Let $\sigma_i = \pm 1$, $i \in \Lambda$ be an Ising spin variable and $\sigma_A = \Pi_{i \in A} \sigma_i$, $A \subset \Lambda$, and let $B$ be any bond on the lattice. ($B$ = two-point nearest-neighbour subset of $\Lambda$ for the Ising model.) Let $-J_B$ be the ferromagnetic interaction associated with any bond $B$ on the lattice. With $\mathscr{B}$ the set of all boundary conditions $b$ on the boundary $\partial\Lambda$ ($\Lambda \cap \partial\Lambda = \phi$) of the finite fixed box $\Lambda$, let each element $b \in \mathscr{B}$ be defined as the subset of sites $\{i_0\} \subset \partial\Lambda$ where $\sigma_{i_0} = -1$ $\forall i_0 \in b$; the other points of $\partial\Lambda \backslash b$ are fixed in the configuration $\sigma = +1$. Now, we can decompose $b$ into maximal connected components $\{C_i\}$, $(i = 1, 2, \ldots, \alpha)$ such that $\cup_{i=1}^{\alpha} C_i = b$ and the $C_i \cup C_j$ are not connected for $i \neq j$. We then consider the dual lattice $\Lambda^*$ to $\Lambda$, whose points are the centres of each unit square of $\Lambda \cup \partial\Lambda$. For every bond $B \subset \Lambda \cup \partial\Lambda$ there is associated a dual bond $B^*$ (orthogonal to $B$) and for each boundary field on $b$ there corresponds a dual path $b^*$ given by $b^* = \{B_i^*\}$, which is the set of bonds $\{B_i^*\}$ dual to $\{B_i\} = \{(i, i_0)\}$ such that $\sigma_i \sigma_{i_0} = -\sigma_i$ ($i \in \Lambda$ and $i_0 \in \partial\Lambda$). Moreover $b^* = \cup_i C_i^*$, where $C_i^*$ is a connected set of bonds $\{B_j^*\}$ along the boundary of the open square $\Lambda^*$. Notice that we have $\Pi_{B_i^* \in C_i^*} B_i^* = \bar{C}_i^* = (i_1^*, i_2^*)$, where $i_1^*$ and $i_2^*$ are the two extreme points of $C_i^*$. We now use a low-high temperature duality transformation; it is known that to every pair of dual bonds $(B, B^*)$, the interactions are related by $\tanh K_{B^*}^* = \exp(-2K_B)$, $K_B = \beta J_B$, $\beta = (kT)^{-1}$. The Hamiltonian $H_\Lambda$ on $\Lambda$ is given by $H_\Lambda = -\Sigma_B J_B \cdot \sigma_B$ (boundary fields included). We set $Z_b = \mathrm{Tr} \exp(-\beta H_\Lambda)$, so that $Z_b$ is the partition function with $b$ the boundary condition. By the low-high temperature duality transformation considered above we have that (Gruber *et al* 1977)

$$\frac{Z_{b,\Lambda}}{Z_{\phi,\Lambda}} = \frac{Z_{b,\Lambda}}{Z_{+,\Lambda}} = \langle \sigma_b^* \rangle_{\Lambda^*}(K^*) = \left\langle \prod_{i=1}^{\alpha} \sigma_{\bar{C}_i^*} \right\rangle_{\Lambda^*} (K^*). \tag{1}$$

We now work with high-temperature graphs associated to $\Lambda^*$ with an open boundary condition, which are the low-temperature closed multipolygons originating from the + boundary condition on $\Lambda$. The Sherman theorem on paths (established for an open domain $\Lambda^*$ with arbitrary two-body interactions) applied to $Z_{\Lambda^*}$ states that the reduced partition function $\bar{Z}_{\Lambda^*}$ is given by the exponential of a sum of closed connected trajectories (1-cycles). The cluster expansion becomes

$$\bar{Z}_{\Lambda^*} = \frac{Z_{\Lambda^*}}{2^{|\Lambda^*|}\,\Pi_{B^*\subset\Lambda^*}\cosh K^*_{B^*}} = \exp\left(\sum_{P} W(P)\right)$$

$$= \exp\left(\sum_{P} \frac{(-1)^{Nc}}{\mu P} \prod_{B^*\in P} (\tanh K^*_{B^*})^{n_{B^*}}\right). \tag{2}$$

In equation (2), $P$ denotes any 1-cycles or closed connected path on $\Lambda^*$ by weight $W(P)$. $\mu_P$ is the multiplicity of the 1-cycle $P$, $N_c$ the number of self-crossings of $P$ and $n_{B^*}$ the number of times the bond $B^*$ occurs in $P$. Further $\Sigma_{B^*\in P}\, n_{B^*} = l(P)$ is the length of $P$. Notice that a change of $\pi$ degrees in the trajectory of the path $P$ is not allowed; moreover $\tanh K^*_{B^*} = \exp(-2K_B)$ is the low-temperature Boltzmann factor associated with any bond $B^* \subset \Lambda^*$; the above series converges up to the critical point given by $e^{-2K} = \tanh K$ in the case where $K_B = K, \forall B$. We now proceed and generalise the Sherman theorem on paths to any boundary condition $b$.

*Theorem.* Let $Z_{\Lambda,b}$ be the exact partition function of the two-dimensional Ising model in zero field with arbitrary boundary condition $b$ on $\Lambda$ and let $b^* = \bigcup_{i=1}^{\alpha} C_i^*$ be defined as above. Let $S_{\Pi,\bar{C}_i^*}$ be the set of all trajectories on $\Lambda^*$ (open in $\partial b^* = \partial(\bigcup_i \bar{C}_i^*) = \Pi_i C_i^*$, so that $\partial b^*$ are the endpoints of $b^*$) by total weight $W_{S_{\Pi,\bar{C}_i^*}}$. Then the cluster expansion for $Z_{\Lambda,b}$ on $\Lambda$ is given by

$$Z_{\Lambda,b} = Z_{\Lambda,+}\langle\sigma_b^*\rangle = Z_{\Lambda,+}R_{\Lambda,b} = Z_{\Lambda,+} \sum_{\substack{\cup b_i^* = b^* \\ b_i^* \in b^*}} \prod_i W_{S_{\bar{b}_i^* = \Pi_{K,C_K^*\in b_i^*}\bar{C}_K^*}} \tag{3}$$

where $W_{S_{\bar{C}_i^*}}$ is the sum of the weights of all trajectories $P$ and $\Lambda^*$, open in $\partial C_i^*$ (which start and which end in the two extreme points of $C_i^*$) i.e.

$$W_{S_{\bar{C}_i^*}} = \sum_P (-1)^{Nc} \prod_{B^*\in P} \exp(-2K \cdot n_{B^*}).$$

Thus $Z_{\Lambda,b}$ splits into the product of $Z_{\Lambda,+}$ (up to a factor, the exponential of the sum of all closed connected paths) and $R_{\Lambda,b}$ (the contribution of all open paths, containing the effect of the boundary condition $b$).

*Proof.* We introduce a path $p_i^*$ of auxiliary bonds $\{B_{il}^*\}$ outside $\Lambda^*$ with two-body interactions $\{\chi_{il}\}$ for any component $C_i^*$ of $b^*$ (the construction used in the proof is illustrated in figure 1) and define

$$Z_{\Lambda^*}(\{\lambda_{il}, K^*\}) = \mathrm{Tr}_\sigma \exp\left(-\beta^* H_{\Lambda^*} + \sum_{i,l} \lambda_{il}\sigma_{B_{il}^*}\right) = A\exp\left(\sum_{\bar{P}} W(\bar{P})\right) \tag{4}$$

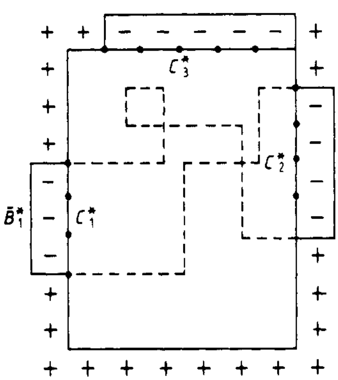


**Figure 1.** A boundary condition $b$, with $b^* = C_1^* \cup C_2^* \cup C_3^*$, the auxiliary path $p_1^* = B_1^*$ for $C_1^*$ and a path appearing in $S_{\bar{C}_1^*\bar{C}_2^*}$.

where $A$ is an immaterial factor and $\bar{P}$ is still any *closed* path which may or may not contain auxiliary bonds $B^*_{il}$ (Sherman theorem). The correlation of interest in equation (3) is then given by

$$\langle\sigma_{b^*}\rangle_{\Lambda^*} = \frac{Z_{\Lambda,b}}{Z_{\Lambda,+}} = \prod_{i,l} \frac{\partial_{\lambda_{il}} Z_{\Lambda^*}(\{\lambda_{il}, K^*\})}{Z_{\Lambda^*}(\{\lambda_{il}, K^*\})}\bigg|_{\{\lambda_{il}\}=0}. \tag{5}$$

Let us first treat the case where $b^*$ just consists of one component, i.e. $b^* = C^*_1$. From the Sherman theorem it is sufficient to assimilate the auxiliary bonds $\{B^*_{1l}\}$ to a single 'bond' $\tilde{B}^*_1$ (a line) with interaction $\lambda_1$, outside $\Lambda^*$, whose endpoints coincide with the endpoints of $c^*_1$, since any path $\bar{P}$ passing through $\{B^*_{1l}\}$ contains every auxiliary bond $B^*_{1l}$ the same number of times. We then have

$$\begin{aligned}\langle\sigma_{\bar{C}^*_1}\rangle_{\Lambda^*}(\{\lambda_1, K^*\}) &= \frac{Z_{\Lambda^*}(\{\lambda_1, K^*\})}{Z_{\Lambda^*}(\{\lambda_1, K^*\})} \\ &\quad\times\left[\tanh\lambda_i + \sum_{\bar{P}\ni\tilde{B}^*_1} \frac{(-1)^{N_{\bar{P}}}}{\mu_{\bar{P}}} \frac{(\tanh\lambda_1)^{n_{\tilde{B}^*_1}-1}}{\cosh^2\lambda_1} \prod_{B^*\in\bar{P}} \exp(-2K_B n_{B^*})\right].\end{aligned} \tag{6}$$

As $\lambda_1 \to 0$ the only contribution is given by those $\bar{P}$ such that $n_{\tilde{B}^*_1} = 1$; thus $\mu_{\bar{P}} = 1$ and we obtain

$$\langle\sigma_{\bar{C}^*_1}\rangle_{\Lambda^*}(\{\lambda_1 = 0, K^*\}) = \sum_{\bar{P}\text{ open in }\bar{C}^*_1} W(\bar{P}) = W_{S_{\bar{C}^*_1}}. \tag{7}$$

Equation (7) proves the theorem for $b^* = C^*_1$, for all $\Lambda$ finite. If $b^* = (C^*_1, C^*_2)$, it is easily seen in the same manner that

$$\langle\sigma_{\bar{C}^*_1}\sigma_{\bar{C}^*_2}\rangle(\{\lambda_1 = \lambda_2 = 0, K^*\}) = W_{S_{\bar{C}^*_1\bar{C}^*_2}} + W_{S_{\bar{C}^*_1}} W_{S_{\bar{C}^*_2}} \tag{8}$$

where $W_{S_{\bar{C}^*_1\cdot\bar{C}^*_2}}$ denotes the contribution of all paths with endpoints $\bar{C}^*_1 \cup \bar{C}^*_2$. For $b^* = (C^*_1, C^*_2, C^*_3)$ we obtain

$$\langle\sigma_{C^*_1\cdot C^*_2\cdot C^*_3}\rangle = W_{S_{\bar{C}^*_1\cdot\bar{C}^*_2\cdot\bar{C}^*_3}} + \prod_{i=1}^{3} W_{S_{\bar{C}^*_i}} + \sum_{i<j\neq K} W_{S_{\bar{C}^*_i\cdot\bar{C}^*_j}} W_{S_{\bar{C}^*_K}}.$$

The formula for the general case $b^* = (C^*_1, \ldots, C^*_\alpha)$ is then immediate and the theorem is proven.

As an application, we briefly discuss the grand canonical cluster expansion for the surface tension, and check graphically its convergence to the SOS limit for $T \leqslant T_c$ after proving that it coincides with the canonical cluster expansion.

## 3. Grand canonical cluster expansion for the surface tension

Let $Z_{+,-}$ be the usual partition function of the model with mixed boundary condition i.e. $\sigma_{i_0} = +1\ \forall_{i_0} = (a, b) \in \partial\Lambda$, $b < 0$ and $\sigma_{i_0} = -1$, $b > 0$, for a box of length $L+1$, height $2M$, with horizontal and vertical interactions $J_1$ and $J_2$ respectively. The surface tension is defined as usual by

$$e^{-\tau L} = \frac{Z_{+-}}{Z_{++}} = \langle\sigma_{i_1}\sigma_{i_2}\rangle_{\Lambda^*}(K^*_1, K^*_2) \tag{9}$$

where $i_1 = (-L/2, 0)$, and $i_2 = (L/2, 0)$. By the theorem we have that

$$\langle\sigma_{i_1}\sigma_{i_2}\rangle_{\Lambda^*}(K^*_1, K^*_2) = \sum_{P_{1,2}} (-1)^{N_{P_{1,2}}} \exp(-2K_2 n_2) \exp(-2K_1 n_1) \tag{10}$$

where $n_i$ is the number of interactions in $P_{1,2}$ along the $i$th direction. To make contact with the SOS limit (defined below), it is useful to consider the grand canonical surface tension $\bar{\tau}$ considered some time ago (Gallavotti 1972). One has

$$\Phi = \exp(-\bar{\tau}L) = \lim_{M\to\infty} \sum_{i\in\mathbb{Z}} \frac{Z^i_{L,M,+-}}{Z_{L,M,++}} = \sum_{i\in\mathbb{Z}} \langle \sigma_{i_1}\sigma_i\rangle_{L,\infty}(K^*). \tag{11}$$

In equation (11), $Z^i_{L,M,+-}$ is defined with $+-$ boundary conditions but where on one side the separation line between $+$ and $-$ is at the height $i$, and where the limit $M\to\infty$ has been taken. $\exp(-\bar{\tau}L)$ is the sum of all paths starting at $i_1=(-L/2, 0)$ and ending at any $i=(L/2, i)$, $i\in\mathbb{Z}$ (see figure 2). We first show that as $L\to\infty$, $\bar{\tau}=\tau\ \forall T\leqslant T_c$. To do this, we apply some inequalities (Messager and Miracle-Sole 1977, Schrader 1977, Hegerfelt 1977, Bricmont 1985), valid in particular for the two-dimensional Ising model on the strip $\tilde{\Lambda}=(L,\infty)$, i.e. $\langle\sigma_{i_1}\sigma_i\rangle\leqslant\langle\sigma_{i_1}\sigma_{i_2}\rangle$ and $\langle\sigma_{i_1}\sigma_i\rangle\leqslant\langle\sigma_{i_2}\sigma_i\rangle$ (see figure 2). Then

$$\sum_{i\in\mathbb{Z}}\langle\sigma_{i_1}\sigma_i\rangle\leqslant\langle\sigma_{i_1}\sigma_{i_2}\rangle^{1-\varepsilon}\sum_{i\in\mathbb{Z}}\langle\sigma_{i_2}\sigma_i\rangle^{\varepsilon} \qquad \forall\varepsilon>0.$$

Since for $T>T_c$, $\langle\sigma_{i_2}\sigma_i\rangle\leqslant\exp(-m|i_2-i|)$ for some $m>0$, we obtain $\Sigma_{i\in\mathbb{Z}}\langle\sigma_{i_2}\sigma_i\rangle^{\varepsilon}\leqslant C(\varepsilon,m)$ uniformly in $L$; moreover $\lim_{L\to\infty}L^{-1}\ln C=0$. From the definition of $\bar{\tau}$ we then have that $\bar{\tau}\geqslant(1-\varepsilon)\tau$. On the other hand, application of the Griffith's inequality gives $\Sigma_{i\in\mathbb{Z}}\langle\sigma_{i_1}\sigma_i\rangle\geqslant\langle\sigma_{i_1}\sigma_{i_2}\rangle$. Thus $\bar{\tau}\leqslant\tau$. Finally $\tau(1-\varepsilon)\leqslant\bar{\tau}\leqslant\tau$. By taking the lim $\varepsilon\to0$, we thus obtain $\tau=\bar{\tau}$, $T\leqslant T_c$. So the grand canonical surface tension coincides with the canonical one for $T\leqslant T_c$; this result is clearly not restricted to the two-dimensional Ising model under consideration here where it is also known by explicit computation that $\tau=\tau_0$ where $\tau_0$ is the surface tension calculated by Onsager. So, without knowing that $\tau=\tau_0$, the point to be investigated is the relationship between the SOS limit and the grand canonical surface tension (this would also be of interest for more general models where $\tau$ is not known). The problem may be analysed in a graphical context. For the two-dimensional Ising model the conjecture is that all trajectories having at least four bonds at some point and coming back cancel exactly and thus that for the model, the SOS limit is exact for $T\leqslant T_c$. The fundamental reason why and when the SOS limit is exact is hard to discover using standard inequalities, even with the one discussed above and established in two dimensions for $T\leqslant T_c$. The difficulty persists also even if one tries to solve the problem using symmetry properties of the graphs associated with the cluster expansion. So, in this section, we limit ourselves to a short

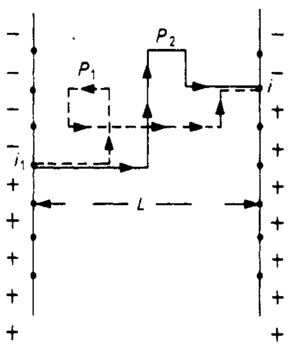


**Figure 2.** Two paths appearing in the cluster expansion of $\langle\sigma_{i_1}\sigma_i\rangle$. $P_1$ compensates exactly for another path $\tilde{P}_1$ not shown in the figure. $P_2$ appears also in the computation of the SOS limit.

discussion and investigate (11) to some order in the expansion parameter by using the cluster expansion. With $\Phi' = \exp[2K_2(L-1)]\Phi$ the SOS limit for the strip $(L, \infty)$ is defined and easily computed to be

$$\Phi'_{\mathrm{SOS}}(K_1) = \lim_{K_2\to\infty} \Phi'(K_1, K_2) = \exp(-L \ln \tanh K_1). \tag{12}$$

$\Phi'$ may also be computed for $K_2 \to 0$. In fact

$$\Phi'(K_1, K_2\to 0) = \sum_{i\in\mathbb{Z}} \frac{Z^i_{+-}(K_1, K_2=0)}{Z_{++}(K_1, K_2=0)} = 1 + 2\sum_{i=1}^{\infty} \left(\frac{\bar{Z}_{+-}}{\bar{Z}_{++}}\right)^i \tag{13}$$

where $\bar{Z}_{+-}$ and $\bar{Z}_{++}$ are the partition functions of a one-dimensional Ising model of shape $L$ with interaction $J_1$ and boundary conditions $+-$ and $++$ respectively. Then

$$\Phi'(K_1, K_2=0) = 1 + 2\frac{\bar{Z}_{+-}}{\bar{Z}_{++}}\left(1 - \frac{\bar{Z}_{+-}}{\bar{Z}_{++}}\right)^{-1}. \tag{14}$$

Since

$$\bar{Z}_{+-}/\bar{Z}_{++} = (1-x^L)/(1+x^L)$$

where $x = \tanh K_1$, we have that

$$\Phi'(K_1, K_2=0) = x^{-L} = \exp(-L \ln \tanh K_1) = \Phi'(K_1, K_2=\infty) = \Phi'_{\mathrm{SOS}}. \tag{15}$$

The above equality indicates that if

$$2K_2 - \bar{\tau}(K_1, K_2) = L^{-1} \ln \Phi'(K_1, K_2)$$

were monotonic in $K_2$, this would imply with (11) that $\tau_{\mathrm{SOS}} = \tau_{\mathrm{Ising}} = 2(K_2 - K_1^*)$ for $T \leqslant T_c$, i.e. for

$$K_2 - K_1^* = K_2 + \tfrac{1}{2} \ln \tanh K_1 \geqslant 0.$$

The first few terms of the cluster expansion for $\Phi$ are easily computed, and are

$$\Phi = \sum_{i\in\mathbb{Z}} \langle\sigma_{i_1}\sigma_i\rangle \cong \exp[-2K_2(L-1)](1 + 2Lx_1 + 2L^2x_2^2 + (\tfrac{4}{3}L^3 + \tfrac{2}{3}L)x_1^3 + \ldots) \tag{16}$$

where $x_1 = e^{-2K_1}$, and this coincides exactly with the expansion of the function

$$\Phi = \exp[-2K_2(L-1)]\left(\frac{1+x_1}{1-x_1}\right)^L$$

which is the SOS limit computed above. Finally, for a one-dimensional strip $(L=2)$, the cluster expansion may easily be computed to higher orders and we obtain

$$\Phi_{L=2} \cong e^{-2K_2}(1 + 4x_1 + 8x_1^2 + 12x_1^3 + 16x_1^4 + 20x_1^5 + 24x_1^6 + \ldots). \tag{17}$$

The series also coincides with the expansion of $\Phi$ above for $L=2$. In § 4 we give a proof that $\bar{\tau} = \tau_{\mathrm{SOS}}$, using the combinatorial method we have worked out; the result is obtained by a careful analysis of the eigenvalues of the matrix propagator in Fourier space.

## 4. Proof of $\bar{\tau} = \tau_{\mathrm{SOS}}$

We now proceed to prove the equality $\bar{\tau} = \tau_{\mathrm{SOS}}$ for $T \leqslant T_c$ using the combinatorial method, which reflects the local structure of the interaction on the lattice; the counting problem for the trajectories is obtained by means of the propagator for trajectories, which in Fourier space is given (Gruber *et al* 1977) by the $4\times4$ matrix:

$$M_{\mu\mu'}(\boldsymbol{K})=\begin{pmatrix} x\,\mathrm{e}^{-\mathrm{i}K_1} & x\,\mathrm{e}^{-\mathrm{i}K_2-\mathrm{i}\pi/4} & 0 & x\,\mathrm{e}^{\mathrm{i}K_2+\mathrm{i}\pi/4} \\ x\,\mathrm{e}^{-\mathrm{i}K_1+\mathrm{i}\pi/4} & x\,\mathrm{e}^{-\mathrm{i}K_2} & x\,\mathrm{e}^{\mathrm{i}K_1-\mathrm{i}\pi/4} & 0 \\ 0 & x\,\mathrm{e}^{-\mathrm{i}K_2+\mathrm{i}\pi/4} & x\,\mathrm{e}^{\mathrm{i}K_1} & x\,\mathrm{e}^{\mathrm{i}K_2-\mathrm{i}\pi/4} \\ x\,\mathrm{e}^{-\mathrm{i}K_1-\mathrm{i}\pi/4} & 0 & x\,\mathrm{e}^{\mathrm{i}K_1+\mathrm{i}\pi/4} & x\,\mathrm{e}^{\mathrm{i}K_2} \end{pmatrix}$$

where $\boldsymbol{K}=(K_1, K_2)$ and $X=\tanh K^*=\mathrm{e}^{-2K}$ (for simplicity we restrict ourselves to the case where $J_1=J_2$). Then, from equation (11) we obtain:

$$\begin{aligned}\Phi&=\int_0^{2\pi}\mathrm{d}K_1\,\mathrm{Tr}\sum_{l=L-1}^{\infty}M^{l}_{\mu\mu'}(K_1, K_2=0)\exp[\mathrm{i}K_1(L-1)]\\ &=\int_0^{2\pi}\mathrm{d}K_1\exp[\mathrm{i}K_1(L-1)]\,\mathrm{Tr}\left(\frac{M^{L-1}_{\mu\mu'}}{I-M_{\mu\mu'}}\right)(K_2=0)\\ &=\int_0^{2\pi}\mathrm{d}K_1\sum_{i=1}^{4}\frac{\lambda_i^{L-1}}{1-\lambda_i}=\int_0^{2\pi}\mathrm{d}K_1\frac{f(\{\lambda_i\})}{\mathrm{Det}[(I-M)(K_1, K_2=0)]}\end{aligned}$$

where

$$\mathrm{Det}(I-M)(K_2=O)=(1+x^2)^2=-2x(1-x^2)(1+\cos K_1)$$

since the eigenvalue equation $M=\lambda I$, i.e. $\mathrm{Det}(M-I\lambda)=0$, is given by the solution of:

$$(\lambda^2+x^2)^2-2\lambda x(\lambda^2-x^2)(1+\cos K_1)=0.$$

We now remark that the matrix propagator of trajectories for the SOS model reduces to

$$M_{\mu\mu'\mathrm{SOS}}(\boldsymbol{K})=\begin{pmatrix} x\,\mathrm{e}^{-\mathrm{i}K_1} & x\,\mathrm{e}^{-\mathrm{i}K_2-\mathrm{i}\pi/4} & 0 & x\,\mathrm{e}^{\mathrm{i}K_2+\mathrm{i}\pi/4} \\ x\,\mathrm{e}^{-\mathrm{i}K_1+\mathrm{i}\pi/4} & x\,\mathrm{e}^{-\mathrm{i}K_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x\,\mathrm{e}^{-\mathrm{i}K_1-\mathrm{i}\pi/4} & 0 & 0 & x\,\mathrm{e}^{\mathrm{i}K_2} \end{pmatrix}.$$

The corresponding eigenvalue equation is given by

$$\mathrm{Det}(M-\lambda I)(K_2=0)=\lambda(\lambda-x)[\lambda^2-\lambda x(1+\mathrm{e}^{-\mathrm{i}K_1})-x^2\,\mathrm{e}^{-\mathrm{i}K_1}]=0.$$

Introducing the variables $\lambda/x=\xi$ and $\mathrm{e}^{\mathrm{i}K_1}=z$, then the eigenvalue equations for the two models are given by

$$\left(z+\frac{\xi(1-\xi)}{1+\xi}\right)\left(z+\frac{1+\xi}{\xi(1-\xi)}\right)=0 \tag{18}$$

$$\xi(\xi-1)\left(z+\frac{1+\xi}{\xi(1-\xi)}\right)=0. \tag{19}$$

For the SOS model (19), $\xi_1=0$ ($\lambda_1=0$) and $\xi_2=1$ ($\lambda_2=x$) do not give any contribution to the integral for $\Phi$. For the Ising model (18), $\xi_1$ and $\xi_2$ are the solutions of

$$z+\xi(1-\xi)/(1+\xi)=0.$$

By a variable change, it may easily be shown that their contribution to $\Phi$ also vanishes. In fact

$$\begin{aligned}\Phi&=\int_0^{2\pi}\mathrm{d}K_1\exp[\mathrm{i}K_1(L-1)]\left(\frac{\lambda_1^{L-1}}{1-\lambda_1}+\frac{\lambda_2^{L-1}}{1-\lambda_2}\right)\\ &=\int_0^{2\pi}\mathrm{d}K_1\exp[\mathrm{i}K_1(L-1)]\left(\frac{\xi_1^{L-1}}{1-x\xi_1}+\frac{\xi_2^{L-1}}{1-x\xi_2}\right).\end{aligned}$$

With $z = e^{iK_1}$ and from $z + \xi(1-\xi)/(1+\xi) = 0$, it is easily seen that as $z$ runs from 0 to $2\pi$, $\xi_1$ runs on the half circle of radius $\sqrt{2}$ with centre $C = (1, 0)$ from the point $P_1 = (1+\sqrt{2}, 0)$ to the point $P_2 = (1-\sqrt{2}, 0)$ and $\xi_2$ from $P_2$ to $P_1$. So the total contribution of $\xi_1$ and $\xi_2$ to $\Phi$ reduces to the contour integral on the above circle of a meromorphic function with no pole inside the circle for $T < T_c$, i.e. $|\xi| < \sqrt{2} - 1$, and

$$\Phi = \int_{\partial C_{\sqrt{2}}} \frac{d\xi}{i} (\xi^2 + 2\xi - 1) x^{L-1} \frac{\xi^{L-1}(1-\xi)^{L-1}}{(1+\xi)^{L+1}(1-x\xi)} = 0.$$

We now remark that $\xi_3$ and $\xi_4$ are identical for the Ising and the SOS model since they are the solutions of the same equation given by $z + (1+\xi)/\xi(1-\xi) = 0$. So this shows that the value of $\Phi$ is the same, reflecting the strong cancellation in the cluster expansion and the proposition is proven.

Clearly, the explicit computation for the contribution of $\xi_3$ and $\xi_4$ to $\Phi$ may be carried out explicitly as for $\xi_1$ and $\xi_2$. We obtain, as before, a contour integral over $\partial C_{\sqrt{2}}$ of a meromorphic function of $\xi$. The difference is the presence of a simple pole at $\xi = 0$ and a pole of order $L$ at $\xi = 1$, i.e.

$$\Phi = \int_{\partial C_{\sqrt{2}}} d\xi \frac{-1}{i} \frac{(\xi^2 + 2\xi - 1)}{\xi} \frac{(1+\xi)^{L-2}}{(1-\xi)^L}.$$

The computation is easier in the $z$ plane than in the $\xi$ plane; for $L > 1$ there is only one simple pole at $z_c = x(1+x)/(1-x)$ for $T < T_c$. We obtain:

$$\begin{aligned}\Phi &= \int_{\partial C_1} dz \frac{z^{L-1}}{i} x^{L-1} \frac{\xi_3^{L-1}(1-x\xi_4) + \xi_4^{L-1}(1-x\xi_3)}{z(1-x) - x(1+x)} \\ &= \int_{\partial C_1} dz \frac{f(z)}{z(1-x) - x(1+x)}.\end{aligned}$$

Notice that $f(z)$ is a meromorphic function of $z$ as may easily be checked. Since

$$\lim_{z \to z_c} f(z) = z_c^{L-1}(1+x)[1 - 2x^2(1+x)^{-2}]$$

we obtain

$$\Phi \cong x^{L-1}[(1+x)/(1-x)]^L$$

and

$$\bar{\tau} = \lim_{L \to \infty} \frac{-\ln \Phi}{L} = 2(K - K^*)$$

for $T \leqslant T_c$, which is the surface tension calculated by Onsager, and obtained here, within the combinatorial method of paths, as a corollary to our proposition. Thus $\tau = \bar{\tau} = \tau_{\mathrm{SOS}} = \tau_0$ for $T \leqslant T_c$.

## 5. Conclusion

In this work we have extended the combinatorial method for the two-dimensional Ising model (on square, hexagonal and triangular lattices) in order to obtain a rigorous cluster expansion for boundary observables like the surface tension in terms of a well defined set of open random trajectories on the lattice; the method may be pursued for other observables inside the lattice with additional non-trivial modifications and this allows for the possibility of an analysis of some interesting open problems in a combinatorial context (see below). As a second step, we have used some refined inequalities for ferromagnetic systems to prove that in the thermodynamic limit the canonical surface tension coincides with the grand canonical one for $T \leqslant T_c$. This result holds for a large class of two-dimensional ferromagnets and for an interface along a main axis of the lattice. Using the above two propositions we have then proved, using the matrix propagator (a peculiarity of the Ising model) that the sos limit is exact for an interface along one of the axes of the lattice.

The sos limit gives the exact result in the region $T \leqslant T_c$ along a main axis since from the above equivalence between the canonical and grand canonical surface tension and from the combinatorial structure of the cluster expansion, any path in the grand canonical surface tension which starts at a given point on one side of the boundary is allowed to reach any point of the other boundary of the string. This kind of allowed fluctuation allows a path with self-crossing occurring in the canonical surface tension (and not compensating for another one) to compensate for a path appearing in the grand canonical surface tension, due to the fact that for an interface along one axis of the lattice, an axial symmetry applied to the path transforms the path into another one excluded from the sos limit but present in the grand canonical expansion. The two paths compensate one for the other since they have the same Boltzmann factor in absolute value and the difference is only a factor $-1$ related to the number of self-crossing modulo 2. Thus only paths which contribute to the sos limit persist and all have positive weight. This is exactly what we have proved. A simple example is given in figure 3.

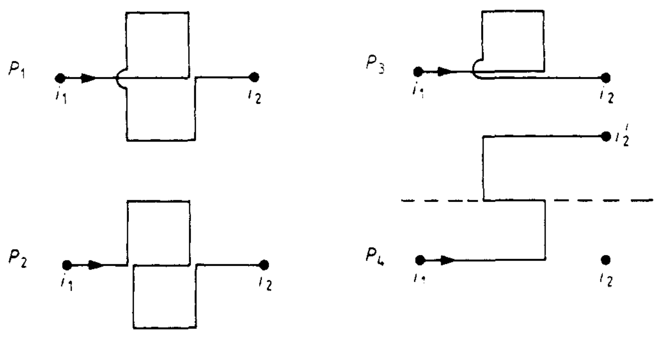


**Figure 3.** Four paths occurring in the grand canonical cluster expansion; $P_1$ compensates for $P_2$ in the canonical expansion. $P_3$ compensates for $P_4$ only in the grand canonical expansion.

From the above discussion and propositions we expect that the result holds only for an interface along one of the main axes. One of the reasons is that the equality between canonical and grand canonical surface tension may not be proven using the refined inequalities. The equality $\tau = \tau_{\mathrm{SOS}}$ is nevertheless expected to be correct for an interface along a main axis for a larger class of two-dimensional ferromagnets. For example, in the two-dimensional Ising model on a triangular lattice with *three-body* nearest-neighbour interactions (a model which is self-dual and which has the same critical point as the Ising model considered here) it is known that $\tau \to 0$ as $|T - T_c|^{2/3}$, $\nu = \frac{2}{3}$, instead of 1; the equality $\tau_c = \tau_{gc}$ may be proven. We expect that $\tau = \tau_{\mathrm{SOS}}$; here this equality may not be proven due to the lack of a matrix propagator for trajectories.

We then expect that the more general method of the transfer matrix could be more useful to solve the general problem (due to the existence of some symmetry properties in the transfer matrix which have not yet been discovered) and may allow us to extract just the SOS contribution. This would be of interest in a geometrical computation of the indices $\nu$ in the SOS limit alone. Returning finally to the computations presented in this work, it should nevertheless be mentioned that an extension of the method of paths given here may be of significance in the search for an exact solution of the susceptibility in closed form for the Ising model. In fact preliminary computations yield the first few terms of the susceptibility series exactly; if a local weight in the counting problem of the paths in the susceptibility series may be obtained, then the susceptibility could be given in terms of integrals over elliptic functions as discussed some time ago (Temperley 1972).

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